Int. J. Heat Mass Transfer. Vol. 19, pp. 175–184. Pergamon Press 1976. Printed in Great Britain

WAVE CONCEPTS IN THE THEORY OF HEAT

V. A. BUBNOV
Machine-Building Institute, Moscow, U.S.S.R.

(*Received* 1 *October* 1974)

Abstract—The known fundamental solution of the heat-conduction equation in the form of Poisson's integral is obtained by summing-up particular similarity solutions derived with the help of a similarity group. As these similarity solutions do not satisfy the initial data, then the integral condition is obtained from which the arbitrary constant of a general solution is determined. Then the solution in the form of Poisson's integral is shown to satisfy approximately the initial condition in the form of the discontinuous step function approximating the initial function. This is one of the drawbacks of the fundamental solution of the classical heat-conduction equation.

The paradox on an infinite "heat propagation velocity" is discussed.

Analysing the isotherm behaviour and using Green's theorem, the general non-linear wave equation is derived here in which the speed of isotherm propagation along the normal is used as an experimental parameter. The relationship between this speed and thermal diffusivity is shown. The different particular cases of this wave equation are studied. Some wave equation solutions are shown to correspond to those of the non-linear parabolic equation.

Then, the derivation is given of the wave heat conduction equation from the point of view of molecular kinetic considerations. The conception on time relaxation is generalized using the Maxwell method and taking into account the correlation between components of a heat velocity of atoms of molecules.

NOMENCLATURE

T,	temperature;
t,	time;
x, y, z,	Cartesian coordinates;
K,	thermal diffusivity;
C_v,	isobaric heat capacity;
ρ,	density;
λ,	thermal conductivity;
ξ, η, ζ,	thermal velocities of atoms and molecules;
R,	universal gas constant.

1. ON INCOMPLETENESS OF THE FUNDAMENTAL SOLUTION TO THE HEAT-CONDUCTION EQUATION

CONSIDER the problem on heat propagation in an infinite region. One-dimensional statement of the problem requires integration of the following partial equation

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} \tag{1.1}$$

at the boundary

$$x = -\infty \quad T = 0, \qquad x = +\infty \quad T = 0 \tag{1.2}$$

and initial

$$t = 0, \qquad T = \varphi(x) \tag{1.3}$$

conditions.

It can easily be shown that equation (1.1) is invariant with respect to group transformations governed by the following infinitesimal operators

$$X_1 = T\frac{\partial}{\partial T} + 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial T}.$$

We shall consider the solution governed by the operator X_1. It is of the form

$$T = t^{-1/2}f(\eta), \qquad \eta = \frac{x^2}{t}. \tag{1.4}$$

This solution is also referred to as a similarity solution, since the operator X_1 determines the similarity group which can be found through the analysis of dimensions based on the so-called π-theorem.

Transition from two independent variables x and t to one η is equivalent to reduction of the number of variables in the space of variables T, x, t. This is why boundary conditions (1.2) and (1.3) are not invariant with respect to the group of transforms governed by the operator X_1.

Really, boundary conditions (1.2) in the new space of variables will be

$$\eta = \infty, \qquad f = 0. \tag{1.2a}$$

It will be seen later that boundary condition (1.2a) is sufficient for integration of the ordinary differential equation obtained upon substitution of (1.4) into (1.1). Initial equation (1.3) is therefore unnecessary since the point $t = 0$ transits into the point $\eta = \infty$, and what is more, the form of solution (1.4) does not allow the function $\varphi(x)$ to be found in it.

Substitution of (1.4) into (1.1) gives ordinary differential equation to determine the function f

$$\eta(4Kf'' + f') + \tfrac{1}{2}(4Kf' + f) = 0. \tag{1.5}$$

With regard for boundary conditions (1.2a) its solution takes the form

$$f = A\,e^{-\eta/4K}.$$

Now the solution of (1.1) is re-written as

$$T(x,t)=\frac{A}{\sqrt{t}}\,\mathrm{e}^{-\eta/4K}=\frac{A}{\sqrt{t}}\,\mathrm{e}^{-x^2/4Kt}. \qquad (1.6)$$

The last formula includes an arbitrary constant A which, strictly speaking, should be determined from the initial condition. But, as was pointed out above, in the similarity solutions, the function $\varphi(x)$ cannot be found at $t = 0$. So, the use of initial equation (1.1) gives an additional integral condition for determination of the constant A. This approach is widely used when calculating viscous jets.

Re-write (1.1) as

$$C_v\rho\frac{\partial T}{\partial t}=\lambda\frac{\partial^2 T}{\partial x^2}$$

and integrate it over the whole range of the variable x

$$\int_{-\infty}^{\infty}C_v\rho\frac{\partial T}{\partial t}\,\mathrm{d}x=\lambda\int_{-\infty}^{\infty}\frac{\partial^2 T}{\partial x^2}\,\mathrm{d}x. \qquad (1.7)$$

Assume also that

$$q_x=\lambda\left.\frac{\partial T}{\partial x}\right|_{x=+\infty}=\lambda\left.\frac{\partial T}{\partial x}\right|_{x=-\infty}.$$

Then equation (1.7) turns into

$$\int_{-\infty}^{\infty}C_v\rho T\,\mathrm{d}x=Q=\text{const.} \qquad (1.8)$$

which means that at any time instant the amount of heat within the integration range becomes constant.

Integral condition (1.8) is used to determine the constant A. Now formula (1.6) assumes the final form

$$T(x,t)=\frac{Q\,\mathrm{e}^{-(x^2/4Kt)}}{2C_v\rho\sqrt{(K\pi t)}}$$

Since the operators X_2 and X_3 govern the transfer group over x and t, the last formula may be written as

$$T(x,t)=\frac{Q e^{-\frac{(x-\xi)^2}{4K(t-t_0)}}}{2C_v\rho\sqrt{[K\pi(t-t_0)]}} \qquad (1.9)$$

with ξ_0 and t_0 being some constants.

The question arises how to determine the amount of heat Q. The considerations are as follows. Let at the instant $t = t_0$ the temperature $T = \varphi(x)$, $\varphi(x)$ being a limited function. The constant governs some point at the axis x. In the vicinity of this point we choose some domain of the width $\mathrm{d}\xi$.

Then within $\mathrm{d}\xi$ we substitute the function $\varphi(x)$ for a constant equal to $\varphi(\xi)$. Now compose the product

$$Q_i=C_v\rho\varphi(\xi)\,\mathrm{d}\xi=\text{const}=Q \qquad (1.10)$$

where Q_i is the amount of heat introduced at the instant $t = t_0$ in the vicinity of the point $x = \xi$. Then formula (1.9) governs the temperature in the whole domain, the temperature being affected by a point heat source.

It should be emphasized here that for the infinitesimal difference $t-t_0$ the temperature $T(x, t)$ is finite at rather high values of $(x-\xi)$, i.e. the temperature stabilizes immediately upon the introduction of a point heat source.

This seems to be a paradox, since any physical process possesses some inertia.

It is reported apropos of this that the velocity of heat transfer is infinite. This conclusion cannot be considered perfectly correct since, as we know, none of the authors has defined the concept of heat-transfer velocity.

The above fact is a sound shortcoming of the classical heat conduction equation giving rise to the majority of works trying to eliminate the paradox.

Compose the sum

$$T(x,t)=\sum_i\frac{Q_i\,\mathrm{e}^{-[(x-\xi)^2/4K(t-t_0)]}}{2C_v\rho\sqrt{[K\pi(t-t_0)]}}.$$

As initial equation (1.1) is linear this sum, at the same time, presents a solution. In case of the infinite sum this means that at the instant $t = t_0$ point heat sources of the power Q_i are located along the whole axis $-\infty\leqslant x\leqslant+\infty$.

Within all the intervals $\mathrm{d}\xi$ the function $\varphi(x)$ is substituted by the constant values equal to $\varphi(\xi)$. Taking account of (1.10), the last formula may be re-written as

$$T(x,t)=\sum\frac{\varphi(\xi)\,\mathrm{d}\xi\,\mathrm{e}^{-[(x-\xi)^2/4K(t-t_0)]}}{2\sqrt{[K\pi(t-t_0)]}}.$$

Upon substituting summation for integration, we obtain solution to equation (1.1) in the form of the well-known Poisson integral

$$T(x,t)=\frac{1}{2\sqrt{(K\pi)}}\int_{-\infty}^{\infty}\frac{\varphi(\xi)\,\mathrm{e}^{-[(x-\xi)^2/4K(t-t_0)]}}{\sqrt{(t-t_0)}}\,\mathrm{d}\xi. \qquad (1.11)$$

It is being proved in the text-books of mathematical physics that solution (1.11) satisfies equation (1.1) in a single-valued way. Prove that at $t\to t_0$ temperature $T(x,t)\to\varphi(\xi)$. Under these conditions the value $\gamma=(1/t-t_0)$ will be a major parameter. Then, in accordance with the known asymptotic estimates

$$\int_{-\infty}^{\infty}(\sqrt{\gamma})\varphi(\xi)\,\mathrm{e}^{-\frac{\gamma(x-\xi)^2}{4K}}\,\mathrm{d}\xi\approx\int_{\xi_1}^{\xi_2}\varphi(\xi)(\sqrt{\gamma})\,\mathrm{e}^{-\frac{\gamma(x-\xi)^2}{4K}}\,\mathrm{d}\xi.$$

Let $x-\xi_1>0$ and $x-\xi_2<0$, and $\xi_2-\xi_1$ is commensurable with $\mathrm{d}\xi$. Within $\mathrm{d}\xi$ the function $\varphi(x)$ was approximated by a constant equal to $\varphi(\xi)$. $\varphi(\xi)$ therefore, may be taken from the above integral.

Introduce the new variable

$$\eta=\frac{(\xi-x)\sqrt{\gamma}}{\sqrt{(4K)}}$$

which tends to infinity at $t\to t_0$ and $(x-\xi)\neq 0$. Then

$$T(x,t_0)=\frac{\varphi(\xi)}{\sqrt{\pi}}\int_{\eta_1}^{\eta_2}\mathrm{e}^{-\eta^2}\,\mathrm{d}\eta=\frac{\varphi(\xi)}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\eta^2}\,\mathrm{d}\eta$$
$$=\varphi(\xi).$$

Usually, when satisfying the initial condition, the function $T(x, t_0)$ transits into $\varphi(x)$. In our case, however, $T(x, t)$ tends to $\varphi(\xi)$ that within $\mathrm{d}\xi$ is equal to a constant approximating $\varphi(x)$. To put it bluntly, at $t\to t_0$ solution (1.11) tends to the discontinuity function $\varphi(\xi)$ approximating the smooth function $\varphi(x)$, each hatched area (Fig. 1) being proportional to the amount of heat generated by the point source in the vicinity of the point ξ.

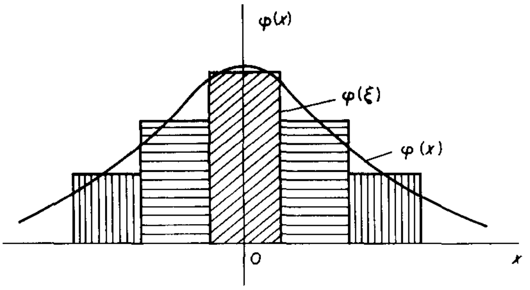

FIG. 1. Approximation of initial function in the Poisson integral.

Such an unusual satisfaction of the initial condition in the fundamental solution should be recognized as one more shortcoming in the classical heat conduction equation.

The above shortcomings may presumably take place only at infinite integration ranges. For a finite domain the solution of equation (1.1) is sought by the variable separation method, and does not possess these defects. It is clear physically since in case of a finite heating region we fail to feel the inertia of the process. This is, presumably, why none of the experimentators measuring a heat-transfer coefficient in finite regions encountered the above paradox.

It is known that derivation of equation (1.1) is first of all connected with the name of Fourier who was the first to apply the mathematical analysis to the theory of heat in 1807. Later, Ohm's theory of electricity flow has been copied from Fourier's theory of heat. At the moment, however, after the works of famous Maxwell wave concepts have been developed in the theory of electromagnetic phenomena. Their mechanical analogues are oscillations of a homogeneous harmonic oscillator described by the following ordinary differential equation

$$m\frac{d^2x}{dt^2}+a\frac{dx}{dt}+bx = F(t). \tag{1.12}$$

Here x is a shift from the equilibrium; m is the oscillator mass; a is the extinction coefficient; ω is angular frequency; $F(t)$ is the compelling force.

The above coefficients have respective electromagnetic analogues.

Comparison of equations (1.1) and (1.12) shows the absence of the second time variable in the classical heat-conduction equation which means the absence of the mass analogue in the theory of heat. Therefore, all the processes described by (1.1) will be inertialess.

The problem arises on the possible ways of improving equation (1.1) in order to eliminate its shortcomings. The simplest way is to take account of the nonlinear term due to temperature dependence of the thermal conductivity.

In [1] similarity solutions are obtained for the power-law dependence of the thermal conductivity on temperature. In these solutions the process of instantaneous heat distribution is somewhat smoothed over depending on the power exponent of the above functional relation. In the case, however, because of the nonlinearity of the initial equation, we cannot compose the sum of partial similarity solutions, and the initial condition cannot be satisfied even in the above sense. Moreover, the experiments show that for fluids, for example, the temperature dependence of the thermal conductivity is weak.

As far back as 1939, Shuleikin noted that in the atmosphere heat is transferred by air masses possessing inertia, and sharp antiphase temperature and pressure fluctuations are accounted for by the generation of thermobaric waves strongly affecting the weather on the Earth. In his opinion the weather changes in the form of thermobaric seiche. In winter time air from the continent flows to the sea close to the ground, and on the top warm air flows from the sea to the continent. These two different flows interact in such a way that in the lower layer temperature conditions obey, in Shuleikin's opinion, the hyperbolic heat-transfer conditions. Shuleikin tried to explain this idea both theoretically and experimentally. To this aim, based on the observations he composed the charts, wherefrom it can easily be seen that the isotherms behave like sound waves. Up to the present time Shuleikin has accumulated vast experimental material proving his ideas [2].

2. ON THE ANALYSIS OF TEMPERATURE FIELD ISOTHERMS

Initial derivations of the hyperbolic heat-conduction equation are connected with the names of Cattaneo [3], Vernotte [4] and Luikov [5]. These authors have used different approaches, but applied to one general idea implying that the heat flux is relaxing, the relaxation time being determined according to Maxwell. Realization of this idea provides the LHS of equation (1.1) with the second time derivative of temperature, the coefficient in this term being exactly equal to the relaxation time which is usually rather small.

This result contradicts Shuleikin's observations for dense atmospheric layers. And what is more, not long ago Antonishin from the Heat and Mass Transfer Institute of the Byelorussian Academy of Sciences superimposed the solution of the hyperbolic heat-conduction equation on experimental heat-transfer data for a surface with a dispersed heat-transfer agent at low Fourier numbers. The agreement between the theory and experiment was fine. But since under these conditions the relaxation time is also small, he substituted it, purely tentatively, by the time of temperature levelling between gas and particles and gave an algorithm to calculate this value [6]. This is one more indication to the existence of wave heat transfer in nature.

The approach suggested by Predvoditelev from the Moscow State University to derive hyperbolic heat-conduction equation is different. He fully refused from composing heat balance when transiting to continuum equations and was the first to apply the Riemann and Helmholtz ideas concerning the theory of manifolds to heat phenomena [7]. He represents manifold of thermal states by the temperature surfaces $T = T(x, y, z, t)$ and thermal conductivity $\lambda = \lambda(x, y, z, t)$. The generation of equal-temperature surfaces moving about the medium involves the onset of heat-transfer process. The velocity of such a surface is defined by the formula

$$g = -\frac{\dfrac{\partial T}{\partial t}}{\dfrac{\partial T}{\partial n}}. \tag{2.1}$$

To understand the formula, we should point it out that the surface velocity is defined as the ratio of the infinitesimal increment of the normal, to infinitesimal time interval. Well then, by using Green's formula and the properties of expanding Riemann's manifold Predvoditelev derives different forms of wave heat-conduction equations.

Predvoditelev's considerations have some limitation concerning the constant velocity g. With the limitation such as that, the diversity of the forms of the heat-conduction equation stems from the diversity of the geometric properties of Riemann's manifold.

By assuming that at each point of the thermal manifold the local heat balance may be calculated by the classical heat-conduction equation, we can easily notice the relation between the speed of propagation of equal-temperature surface and thermal conductivity. Really, by expressing $\partial T/\partial t$ of (2.1) in terms of three-dimensional equation (1.1), we arrive at

$$g = -\frac{\operatorname{div}(\lambda \operatorname{grad} T)}{C_v \rho \dfrac{\partial T}{\partial n}}. \tag{2.2}$$

It follows from this formula that $\lambda = \lambda(x, y, z, t)$ and $g = \text{const}$ is a particular case of thermal manifold.

In a general case it should be considered that $g = g(x, y, z, t)$. Then by narrowing the geometric manifold properties diverse forms of the heat-conduction equation may be obtained due to diversity of the functions of g. In point of fact, these are the same geometrical properties of thermal manifold in an implicit form.

Assume that the temperature field is prescribed for the surfaces of equal temperatures $T(t, n) = \text{const}$ moving in the space at the velocity g. For these surfaces, we have a right to write

$$\mathrm{d}T = \frac{\partial T}{\partial t} + \frac{\mathrm{d}n}{\mathrm{d}t}\frac{\partial T}{\partial n} \equiv 0$$

or, with account for $g = \mathrm{d}n/\mathrm{d}t$, finally obtain

$$\frac{\partial T}{\partial t} + g\frac{\partial T}{\partial n} = 0. \tag{2.3}$$

Differentiation of equation (2.3) over t and n with further elimination of the mixed derivative $\partial^2 T/\partial t \partial n$ from the above equations give

$$\frac{\partial^2 T}{\partial t^2} + \left(\frac{\partial g}{\partial t} - g\frac{\partial g}{\partial n}\right)\frac{\partial T}{\partial n} = g^2\frac{\partial^2 T}{\partial n^2}.$$

Remembering (2.1), express $\partial T/\partial n$ in terms of $\partial T/\partial t$ and obtain

$$\frac{\partial^2 T}{\partial t^2} + \left(\frac{\partial g}{\partial n} - \frac{1}{g}\frac{\partial g}{\partial t}\right)\frac{\partial T}{\partial t} = g^2\frac{\partial^2 T}{\partial n^2}. \tag{2.4}$$

In the equation obtained the right-hand part of it is somewhat unusual since in mathematical analysis the derivatives over the coordinates rather than those over the normals are dealt with. But this shortcoming can be eliminated.

Integrate the left- and right-hand parts of (2.4) over the volume

$$\iiint \frac{\partial^2 T}{\partial t^2}\,\mathrm{d}v + \iiint \left[\left(\frac{\partial g}{\partial n} - \frac{1}{g}\frac{\partial g}{\partial t}\right)\frac{\partial T}{\partial t}\right]\mathrm{d}v = \iiint g^2\frac{\partial^2 T}{\partial n^2}\,\mathrm{d}v \tag{2.5}$$

Then integrate the RHS of (2.5) by parts

$$I = \iiint g^2 \frac{\partial^2 T}{\partial n^2}\,\mathrm{d}v = \iint g^2\frac{\partial T}{\partial n}\,\mathrm{d}s - \iiint 2g\frac{\partial g}{\partial n}\frac{\partial T}{\partial n}\,\mathrm{d}v.$$

In the equality obtained the surface integral may again be expressed through the volume integral using Green's function.

For two functions $\Psi = \Psi(x, y, z, t)$ and $\varphi = \varphi(x, y, z, t)$, the latter is of the form

$$\iint \Psi\frac{\partial \varphi}{\partial n}\,\mathrm{d}s = \iiint \frac{\partial \Psi}{\partial n}\frac{\partial \varphi}{\partial n}\,\mathrm{d}v + \iiint \Psi\nabla^2\varphi\,\mathrm{d}v$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In our case $\Psi = g^2$ and $\varphi = T$. Accordingly, we shall have

$$I = \iiint g^2\frac{\partial^2 T}{\partial n^2}\,\mathrm{d}v = \iiint g^2\nabla^2 T\,\mathrm{d}v. \tag{2.6}$$

Now re-write equation (2.5) in the form

$$\iiint \frac{\partial^2 T}{\partial t^2}\,\mathrm{d}v + \iiint \left[\left(\frac{\partial g}{\partial n} - \frac{1}{g}\frac{\partial g}{\partial t}\right)\frac{\partial T}{\partial t}\right]\mathrm{d}v = \iiint g^2\nabla^2 T\,\mathrm{d}v.$$

By sparing from integrals through usual operation, we obtain the following differential heat-transfer equation

$$\frac{\partial^2 T}{\partial t^2} + \left(\frac{\partial g}{\partial n} - \frac{1}{g}\frac{\partial g}{\partial t}\right)\frac{\partial T}{\partial t} = g^2\nabla^2 T. \tag{2.7}$$

If g dependence on the coordinates and time is implicit, i.e. $g = g(T)$, the latter equation assumes the form

$$\frac{\partial^2 T}{\partial t^2} + \frac{\mathrm{d}g}{\mathrm{d}T}\left(\frac{\partial T}{\partial n} - \frac{1}{g}\frac{\partial T}{\partial t}\right)\frac{\partial T}{\partial t} = g^2\nabla^2 T. \tag{2.8}$$

This is a nonlinear wave equation with the isotherm speed propagation $g = g(T)$ as the only experimental coefficient which is related with thermal conductivity, heat capacity and density of material by formula (2.2). If

$$\frac{\partial T}{\partial t} - g\frac{\partial T}{\partial n} = 0 \tag{2.9}$$

equation (2.8) degenerates into nonlinear wave equation whose form coincides with nonlinear acoustic equation

$$\frac{\partial^2 T}{\partial t^2} = g^2 \nabla^2 T. \tag{2.10}$$

Equation (2.10) may be transformed into a telegraph equation by assuming

$$g^2 T = C^2(T) - A(T)\frac{\dfrac{\partial T}{\partial t}}{\nabla^2 T}. \tag{2.11}$$

Then (2.10) becomes

$$\frac{1}{A(T)}\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{C^2(T)}{A(T)}\nabla^2 T. \tag{2.12}$$

Equation (2.11) can be made more understandable by assuming that the local heat balance may be calculated by linear equation (1.1). Now substitute the term $\partial T/\partial t$ in (2.11) for the RHS of three-dimensional equation (1.1) and obtain

$$\frac{g^2(T)}{A(T)} = \frac{C^2(T)}{A(T)} - K. \tag{2.13}$$

Suppose that the LHS of this equation vanishes at the finite velocity g. This is equivalent to $A(T)$ tending to infinity which results in vanishing of the wave term in equation (2.12) and in the following equation

$$K = \frac{C^2(T)}{A(T)}.$$

This allows the form of (2.12)

$$\frac{1}{A(T)}\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = K\nabla^2 T. \tag{2.14}$$

The thermal diffusivity K for the telegraph equation of such a form may be obtained by usual measurements, and the difference between the classical heat-conduction and experiment may be eliminated through the appropriate choice of the coefficient $A(T)$.

3. ON THE CHARACTERISTICS OF EQUATION (2.10)

Consider the one-dimensional wave equation

$$\frac{\partial^2 T}{\partial t^2} = g_0^2 \frac{\partial^2 T}{\partial x^2} \tag{3.1}$$

being one-dimensional analogue of equation (2.10).

It follows from the above conclusion that two isotherms

$$\left.\begin{aligned} \frac{\partial T}{\partial t} + g_0\frac{\partial T}{\partial x} = 0 \\ \frac{\partial T}{\partial t} - g_0\frac{\partial T}{\partial x} = 0 \end{aligned}\right\} \tag{3.2}$$

that may be regarded as the system of two first-order equations equivalent to (3.1), are appropriate to equation (2.10).

Indeed, differentiation of first equation of (3.2) over x and t and elimination of the mixed derivative $\partial^2 T/\partial x \partial t$ out of the equations obtained gives

$$\frac{\partial^2 T}{\partial t^2} + \left(\frac{\partial T}{\partial t} - g_0\frac{\partial T}{\partial x}\right)\frac{\mathrm{d}g_0}{\mathrm{d}T}\frac{\partial T}{\partial x} = g_0^2\frac{\partial^2 T}{\partial x^2}.$$

Due to the second equation of system (3.2) the parentheses in the last equation vanishes, and equation assumes the form (3.1).

Quasilinear system of equations (3.2) has two solutions

$$\begin{aligned} T_1 &= x - g_0 t = \text{const} \\ T_2 &= x + g_0 t = \text{const} \end{aligned} \tag{3.3}$$

which can easily be verified through simple substitution.

The straight lines of (3.3) are the characteristics of system (3.2), so in this case the families of isotherms and characteristics coincide.

It can easily be shown that when transiting via these lines temperature derivatives suffer discontinuity [8]. This fact controls the conditions when the process of heat propagation is defined by (3.1).

In the wave theory the concept of phase velocity ranks high. It would be of interest to compare this with the concept of speed of propagation.

The equation of one-dimensional running wave is of the form

$$T(x, t) = A\cos(\omega t - kx).$$

Its phase function $\varphi(x, t)$ is determined as an argument of the wave function $\cos(\omega t - kx)$

$$\varphi(x, t) = \omega t - kx.$$

The complete differential of $\varphi(x, t)$ is

$$\mathrm{d}\varphi = \frac{\partial \varphi}{\partial t}\mathrm{d}t + \frac{\partial \varphi}{\partial x}\mathrm{d}x = \omega\,\mathrm{d}t - k\,\mathrm{d}x.$$

From the condition for this equation to be zero, phase velocity is defined as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{K}. \tag{3.4}$$

For the same running wave, the speed of propagation g_0 is governed by the formula

$$g_0 = -\frac{\dfrac{\partial T}{\partial t}}{\dfrac{\partial T}{\partial x}} = \frac{\omega}{k}. \tag{3.5}$$

So, for running waves the velocity g_0 coincides with the phase velocity.

Consider stagnant one-dimensional wave

$$T(x, t) = A(x)\cos \omega t.$$

Its phase function $\varphi = \omega t$, and it is impossible to find its phase velocity $\partial x/\mathrm{d}t$. But it can easily be shown that the speed of its propagation is defined as

$$g_0 = \frac{\omega \tan \omega t}{\dfrac{\mathrm{d}A}{\mathrm{d}x}}.$$

Hence it appears, that a speed of propagation is a more general concept than a phase velocity.

4. GROUP PROPERTIES OF EQUATION (3.1)

The classical heat-conduction equation is known to allow some wave solutions. Parabolic and hyperbolic equations, therefore, should have some properties in common.

We shall try to determine common group properties of the nonlinear heat-conduction equation

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(K(T)\frac{\partial T}{\partial x}\right) \tag{4.1}$$

and of equation (3.1).

Group properties of equation (4.1) have been studied by Ovsyannikov [9]. As for us, we should therefore only verify the invariance of equation (3.2) with respect to group transformations allowed by the heat-conduction equation.

1. With arbitrary $K(T)$ and $g_0(T)$ equation (3.1) and (4.1) are invariant with respect to group transformations determined by the operators

$$X_1=\frac{\partial}{\partial x},\qquad X_2=\frac{\partial}{\partial t}.$$

These operators govern invariant-group solutions of the form

$$T=T(x\pm at) \tag{4.2}$$

i.e. purely wave solutions.

2. If $K(T)$ is one arbitrary function, and $g_0^2=A/t$, then equations (3.1) and (4.1) allow three independent operators

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\quad X_3=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}. \tag{4.3}$$

This means that group properties of equation (4.1) are identical to that of the following wave equation

$$\frac{\partial^2 T}{\partial t^2}=\frac{A}{t}\frac{\partial^2 T}{\partial x^2}. \tag{4.4}$$

3. Let $K(T)=\mathrm{e}^T$ and $g_0^2(T)=\mathrm{e}^T$, equations (3.1) and (4.1) are then invariant with respect to the transformations determined by the operators

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\quad X_4=-t\frac{\partial}{\partial t}+\frac{\partial}{\partial T}. \tag{4.5}$$

4. If $K(T)=T^{2m}$ and $g_0^2(T)=T^{2m}$, equations (3.1) and (4.1) allow operators

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\quad X_5=mx\frac{\partial}{\partial x}+T\frac{\partial}{\partial T}. \tag{4.6}$$

The analysis made indicates to the generality of solutions for parabolic and hyperbolic equations. Hence, understandable is the success of work [1] suggesting an appropriate choice of functional dependence $K=K(T)$ to eliminate the paradox of inertialess heat propagation. And what is more, this can help in explaining the fact, that the main equation of quantum mechanics is a parabolic-type equation but it well describes wave processes.

5. MOLECULAR-KINETIC GROUNDATION OF THE PARABOLIC HEAT-CONDUCTION EQUATION

Maxwell presented a method of derivation of transfer equations for any quantity, which does not imply the calculation of the velocity distribution function of molecules, at all.

The method was used for derivation of the heat-conduction equation by the well-known naturalist Kirchhol and Russian physicist Stankevich [10]. Their calculations almost coincide, but when trying to obtain a parabolic heat-conduction equation, they have made some assumptions whose details differ.

Consider monoatomic gas with the velocity of evident motion equal to zero. In this case heat is transferred only by conduction. If Q is any quality of a molecule, then the Maxwell transfer equation is of the form

$$N\frac{\mathrm{d}Q}{\mathrm{d}t}=-\left[\frac{\partial}{\partial x}(\overline{N\xi Q})+\frac{\partial}{\partial y}(\overline{N\eta Q})+\frac{\partial}{\partial z}(\overline{N\zeta Q})\right]+\Delta Q. \tag{5.1}$$

Here ΔQ is collision integral.

Substitute $Q=m(\xi^2+\eta^2+\zeta^2)$ into (5.1) and, by taking account of the constant kinetic energy at collision, obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(\overline{\xi^2+\eta^2+\zeta^2})]=-\left\{\frac{\partial}{\partial x}[\rho\xi(\overline{\xi^2+\eta^2+\zeta^2})]+\frac{\partial}{\partial y}[\rho\eta(\overline{\xi^2+\eta^2+\zeta^2})]+\frac{\partial}{\partial z}[\rho\zeta(\overline{\xi^2+\eta^2+\zeta^2})]\right. . \tag{5.2}$$

Since the quantity $(\xi^2+\eta^2+\zeta^2)$ is proportional to temperature, the product $\rho\xi(\xi^2+\eta^2+\zeta^2)$ is proportional to the amount of heat per unit area yz. Hence, we may introduce heat fluxes by the formulae

$$\begin{aligned}q_x&=\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}\\ q_y&=\overline{\rho\eta(\xi^2+\eta^2+\zeta^2)}\\ q_z&=\overline{\rho\zeta(\xi^2+\eta^2+\zeta^2)}.\end{aligned} \tag{5.3}$$

Equation (5.2) assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho(\xi^2+\eta^2+\zeta^2)}]=-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}. \tag{5.4}$$

Next we are to calculate heat fluxes q_x, q_y, q_z. Substitute the quantity $m\xi(\xi^2+\eta^2+\zeta^2)$ into (5.1) and obtain

$$m\Delta Q[\overline{\xi(\xi^2+\eta^2+\zeta^2)}]=\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}]+\frac{\partial}{\partial x}[\overline{\rho\xi^2(\xi^2+\eta^2+\zeta^2)}]+\frac{\partial}{\partial y}[\overline{\rho\xi\eta(\xi^2+\eta^2+\zeta^2)}]+\frac{\partial}{\partial z}[\overline{\rho\xi\zeta(\xi^2+\eta^2+\zeta^2)}]. \tag{5.5}$$

Then Stankevich makes an assumption on the absence of friction, i.e.

$$\overline{\xi\eta}=\overline{\xi\zeta}=0$$

and on the steady state of a heat flux which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho\xi(\overline{\xi^2+\eta^2+\zeta^2})]=0. \tag{5.6}$$

Without assumption for (5.6), re-write (5.5) as

$$m\Delta Q[\overline{\xi(\xi^2+\eta^2+\zeta^2)}]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}]+\frac{\partial}{\partial x}[\overline{\rho\xi^2(\xi^2+\eta^2+\zeta^2)}]. \quad (5.7)$$

The LHS of equation (5.7) for maxwellian molecules has been calculated by Boltzmann [11] and is

$$m\Delta Q[\overline{\xi(\xi^2+\eta^2+\zeta^2)}]$$
$$=-2\sqrt{\left(\frac{k}{2m}\right)}\rho^2A_2[\xi(\xi^2+\eta^2+\zeta^2)]. \quad (5.8)$$

Using (5.8), re-write (5.7) as

$$\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}$$
$$=-\frac{1}{2A_2\rho\sqrt{\left(\frac{k}{2m}\right)}}\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}]\right.$$
$$\left.+\frac{\partial}{\partial x}[\overline{\rho\xi^2(\xi^2+\eta^2+\zeta^2)}]\right\}. \quad (5.9)$$

The use of Maxwellian molecule velocity distribution gives

$$\overline{\xi^2(\xi^2+\eta^2+\zeta^2)}=\overline{\xi^4}+\overline{\xi^2\eta^2}+\overline{\xi^2\zeta^2}=5\frac{p^2}{\rho^2}.$$

Now equation (5.9) assumes the form

$$\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}$$
$$=-A_1\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho\xi(\xi^2+\eta^2+\zeta^2)}]-5A_1\frac{\partial}{\partial x}\left(\frac{p^2}{\rho}\right).$$

Here, also

$$A_1=\frac{1}{2A_2\rho\sqrt{\left(\frac{k}{2m}\right)}}. \quad (5.10)$$

If the pressure p does not change from point to point, the latter and similar equations lead to

$$q_x=-A_1\frac{\mathrm{d}q_x}{\mathrm{d}t}-5A_1p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right);$$
$$q_y=-A_1\frac{\mathrm{d}q_y}{\mathrm{d}t}-5A_1p\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right); \quad (5.11)$$
$$q_z=-A_1\frac{\mathrm{d}q_z}{\mathrm{d}t}-5A_1p\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right).$$

Substitution of the above expressions into (5.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho(\xi^2+\eta^2+\zeta^2)}]$$
$$=A_1\frac{\partial}{\partial t}\left[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]+\frac{\partial}{\partial x}\left[5A_1p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[5A_1p\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right)\right]+\frac{\partial}{\partial z}\left[5A_1p\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right].$$

Here the first right-hand bracket is again determined by formula (5.4) and we, finally, obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\overline{\rho(\xi^2+\eta^2+\zeta^2)}]+A_1\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\overline{\rho(\xi^2+\eta^2+\zeta^2)}]$$
$$=\frac{\partial}{\partial x}\left[5A_1p\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)\right]+\frac{\partial}{\partial y}\left[5A_1p\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[5A_1p\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]. \quad (5.12)$$

In accordance with Maxwell's and Boltzmann's concepts, pressure is defined in terms of the thermal velocities as

$$p=\tfrac{1}{3}\overline{\rho(\xi^2+\eta^2+\zeta^2)} \quad (5.13)$$

which again leads to the state equation in the Clayperon form

$$p\equiv\rho RT. \quad (5.14)$$

Since the visible velocities are equal to zero, the continuity equation is reduced to the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\equiv 0$$

which allows the density ρ to be taken from the derivative sign in the LHS of equation (5.12).

Multiplying both sides of equation (5.12) by the specific heat at constant volume C_v and, taking into account (5.13) and (5.14), we arrive at the heat-transfer equation

$$C_v\rho\frac{\mathrm{d}T}{\mathrm{d}t}+C_v\rho A_1\frac{\mathrm{d}^2T}{\mathrm{d}t^2}$$
$$=\frac{\partial}{\partial x}\left(\lambda_0\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_0\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda_0\frac{\partial T}{\partial z}\right) \quad (5.15)$$

where λ_0 is the thermal conductivity equal to

$$\lambda_0(T)=\tfrac{5}{6}\frac{RC_vT}{A_2\sqrt{\left(\frac{k}{2m}\right)}}. \quad (5.16)$$

The use of the relaxation time concept introduced by Maxwell allows the following form of (5.10) [11]

$$A_1=\tfrac{3}{2}\frac{\mu_0}{p_0}=\tfrac{3}{2}\tau_0. \quad (5.17)$$

Here p is equilibrium pressure determined by (5.14).

At atmospheric pressure the relaxation time τ_0 is of the order of 10^{-10} s, i.e. in (5.15) the hyperbolic term at A_1 may be neglected which gives a parabolic heat-conduction equation. The smallness of the relaxation time, in point of fact, justifies the assumption made by Stankevich (5.6).

6. CORRELATING GAS SYSTEMS AND HEAT-CONDUCTION EQUATION

The idea of a correlating gas system, as a logical groundation to Maxwell's concept, has been introduced by Predvoditelev in his works dealing with the state equation for condensed media [12].

The maxwellian velocity distribution of molecules may be regarded as a generalization of Laplace–Gauss one-dimensional distribution function for the case of three statistic signs with no correlation between.

Indeed, find the probability of gas system residence in the phase volume $\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta=\mathrm{d}\omega$. If $n(\xi,\eta,\zeta)$ is equal to the number of molecules whose velocities are within $\mathrm{d}\omega$ and N is the total number of molecules, then the above probability is

$$f(\xi,\eta,\zeta)=\frac{n(\xi,\eta,\zeta)}{N}.$$

Then, denote the probability of molecules resident within the intervals dξ, dη and dζ in terms of $f(\xi)$, $f(\eta)$ and $f(\zeta)$. Assume that these probabilities are independent. Then, in accordance with the theorem of probability product, we have:

$$f(\xi,\eta,\zeta)=f(\xi)f(\eta)f(\zeta). \tag{6.1}$$

Maxwell gives the following groundation to the hypothesis of independent probabilities [13]: "But the existence of the velocity ξ in no way should affect the existence of the velocities η and ζ, since all of them are at right angles to each other and are independent of each other."

Let each of the above probabilities obey the Laplace–Gauss formula

$$f(\xi)=C_1\,\mathrm{e}^{-h\xi^2},\quad f(\eta)=C_2\,\mathrm{e}^{-h\eta^2},\quad f(\zeta)=C_3\,\mathrm{e}^{-h\zeta^2}.$$

Their substitution into (6.1) gives the Maxwellian distribution

$$f=A\,\mathrm{e}^{-h(\xi^2+\eta^2+\zeta^2)}. \tag{6.2}$$

It follows from the above, that the maxwellian distribution of molecule velocities rests upon shaky bases.

But the conclusions in favour of the universality of the above distribution are supported by the fact that it is obtained as one of the solutions to the Boltzmann integro-differential equation. The Boltzmann hypothesis on molecular-irregular state of a gas system is, however, equivalent to the independence of statistic signs which are thermal velocities of atoms and molecules.

In other words, Maxwell's concept (6.1) is present in the Boltzmann equation in an implicit form.

As far back as in 1896, however, the English scientist Burbury pointed out that the presence of rotational degrees of freedom in a gas system may lead to correlation of thermal velocity components [14]. Unfortunately, this idea did not find the followers since the problem of temperature estimation arises in the case and, hence, the principle of uniform energy distribution with respect to the degrees of freedom is violated.

Predvoditelev was the only one who in the above works has introduced the correlation function of molecular velocity distribution into the theory of gases and suggested a method of temperature estimation without violating Boltzmann's postulates. But at the same time he emphasized there that this is not the only way to solve the problem.

Another way to the solution of the above problem can also be suggested.

Indeed, the correlation in thermal velocity components of atoms and molecules leads to the distribution function of the following form [15]

$$f=A\exp\left[-\tfrac{1}{2}\left(\frac{R_{11}}{R}\frac{\xi^2}{\sigma_1^2}+\frac{R_{22}}{R}\frac{\eta^2}{\sigma_2^2}+\frac{R_{33}}{R}\frac{\zeta^2}{\sigma_3^2}\right.\right.$$
$$\left.\left.+2\frac{R_{12}}{R}\frac{\xi\eta}{\sigma_1\sigma_2}+2\frac{R_{13}}{R}\frac{\xi\zeta}{\sigma_1\sigma_3}+2\frac{R_{23}}{R}\frac{\eta\zeta}{\sigma_2\sigma_3}\right)\right]. \tag{6.3}$$

Here R is correlation determinant, σ_1, σ_2, σ_3 are RMS deviations proportional to the kinetic energy over the appropriate degrees of freedom.

Suppose that σ_1, σ_2 and σ_3 are expressed in terms of some quantity σ, whose meaning will be revealed later as follows

$$\sigma_1=\sqrt{\left(\frac{R_{11}}{R}\right)}\sigma;\ \sigma_2=\sqrt{\left(\frac{R_{22}}{R}\right)}\sigma;\ \sigma_3=\sqrt{\left(\frac{R_{33}}{R}\right)}\sigma. \tag{6.4}$$

In case of isotropic correlation

$$R_{11}=R_{22}=R_{33}=1-r^2,$$
$$R_{12}=R_{13}=R_{23}=-r(1-r),\quad R=(1-r)^2(1+2r).$$

Then, formula (6.3) with account of (6.4) assumes the form

$$f=A\,\mathrm{e}^{-\frac{1}{2\sigma^2}\left[\xi^2+\eta^2+\zeta^2-\frac{2r}{1+r}(\xi\eta+\eta\zeta+\xi\zeta)\right]}. \tag{6.5}$$

In mathematical statistics the correlation coefficient r ranges from -1 to $+1$, but the case may be different for a gas system. Extend the range of r through the following transformation

$$\frac{r}{1+r}=\frac{\beta r_1}{1+r_1}.$$

The parameter β will be different for different gases and, in a general case, may be the function of temperature and density. In order to find its numerical values we should apply to the experiment.

Now the molecular velocity distribution function assumes the form

$$f=A\,\mathrm{e}^{-\frac{1}{2\sigma^2}\left[\xi^2+\eta^2+\zeta^2-2\frac{\beta r_1}{1+r_1}(\xi\eta+\eta\zeta+\xi\zeta)\right]}. \tag{6.6}$$

The latter can be reduced to a canonical form through the solution of the third-power equation governed by a determinant

$$\begin{vmatrix} S-1 & -n & -n \\ -n & S-1 & -n \\ -n & -n & S-1 \end{vmatrix}=0.$$

Here

$$n=\frac{\beta r_1}{1+r_1}.$$

The roots of these equations are

$$S_1=1-2n,\qquad S_2=1+n,\qquad S_3=1+n.$$

Now the canonical form of formula (6.6) is

$$f=A\,\mathrm{e}^{-\frac{1}{2\sigma^2}(S_1\xi^2+S_2\eta^2+S_3\zeta^2)}. \tag{6.7}$$

It can easily be understood that if $r_1=0$, $s_1=s_2=s_3=1$ and distribution function (6.7) turns into the maxwellian function, its last part being a particular case.

Calculate the constant A and mean-square velocity of thermal motion $\overline{c^2}$ through ordinary approaches.

$$A=\frac{\sqrt{(S_1S_2S_3)}}{\sigma^3(2\pi)^{3/2}}$$

$$\overline{c^2}=\frac{3(1+r_1)[1+(1-\beta)r_1]\sigma^2}{[1+(2\beta-1)r][1+(1+\beta)r_1]}=3\Psi(r_1\beta)\sigma^2.$$

Assuming no intramolecular forces in accordance with the virial theorem developed by Clausius, we arrive at

$$pv = \tfrac{2}{3}K$$

where K is mean kinetic energy of a molecule of gas of specific volume v. In accordance with our predictions

$$K = \frac{Nmc^{-2}}{2} = \tfrac{3}{2}Nm\sigma^2\Psi(r_1, \beta). \qquad (6.8)$$

Since N is the Avogardo number, the quantity σ may be related with temperature after Boltzmann's postulate $Nm\sigma^2 = RT$.

Now equation (6.8) may be re-written as

$$pv = \Psi(r_1, \beta)RT. \qquad (6.9)$$

If $r_1 = 0$, the latter equality transits into the Clayperon equation.

To solve the above stated problem, we shall interpret formula (6.9) as follows.

Suppose that heat is transferred frontally. Then either chemical reactions or phase conversions may proceed on the front surface, and the thermal conductivity determined by (5.16) will fail to level temperature nonuniformities.

We have a right to suppose for such a system that pressure, density and temperature are related by equation (6.9), which is re-written as

$$p = p_0\Psi(r_1, \beta) \qquad (6.10)$$

where p_0 is the equilibrium pressure determined by Claypeyron formula (5.14).

Equation (6.10) should be interpreted as an equation of nonequilibrium process that controls the pressure excess over equilibrium one and results from the processes other than heat transfer only by conduction.

The use of formula (6.10) for transition from (5.12) to (5.15) gives

$$C_v\rho\frac{\mathrm{d}T}{\mathrm{d}t} + C_v\rho A_1\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right) \qquad (6.11)$$

where $\lambda(T) = \Psi(r_1, \beta)\lambda_0(T)$.

In order to correctly define the quantity A_1, in this case the pressure p in (5.17) should be determined by (6.10), i.e.

$$A_1 = \tfrac{3}{2}\frac{\mu_0}{p_0\Psi} = \tfrac{3}{2}\frac{\tau_0}{\Psi(r_1, \beta)}. \qquad (6.12)$$

Now, in spite of the smallness of the quantity τ_0, the coefficient A_1 will be finite and, in a general case, may be a function of temperature.

It is easily understood that equality (6.14) reduces to equation (2.14) which is obtained phenomenologically.

As to the case of molecular-kinetic grounds for general wave equation (2.8), the derivations we have made should be considered incomplete.

Indeed, introduction of the correlation function of molecular velocity distribution is equivalent to the effect of hygroscopic forces [15]. Therefore, to derive wave equations of heat transfer, mechanical motions similar to latent motions of moving media electrodynamics are necessary.

We have presented only the beginning of the above derivations.

REFERENCES

1. Ya. B. Zeldovich and A. S. Kompaneetz, *To the Theory of Heat Distribution at Temperature-Dependent Thermal Conductivity*. (Coll. papers devoted to A. F. Ioffe) pp. 61–71. Izd. Akad. Nauk SSSR, Moscow (1950).
2. V. V. Shuleikin, *Sea Physics*. Izd. Nauka, Moscow (1968).
3. C. Cattaneo, Sulla conduzione del calore, Societta Tipografica Modenese (1948).
4. P. Vernotte, *C.R. Hebd. Seanc. Acad. Sci., Paris* No. 22, 246 (1958).
5. A. V. Luikov, Methods of irreversible thermodynamics as applied to heat and mass transfer investigation, *Inzh. Fiz. Zh.* **9**, 3 (1965).
6. N. V. Antonishin, M. A. Geller and A. L. Parnas, Hyperbolic heat conduction equation for disperse systems, *Inzh. Fiz. Zh.* **26**, 3 (1974).
7. A. S. Predvoditelev, Heat teaching and Riemann's manifolds, heat and mass transfer problem (Coll. papers). Izd. Energiya (1970).
8. A. N. Tikhonov and A. A. Samarsky, *Mathematical Physics Equations*. Moscow (1972).
9. L. V. Ovsyannikov, Group properties of nonlinear heat conduction equations, *Dokl. Akad. Nauk SSSR* **125**, 3 (1959).
10. B. V. Stankevich, *Mathematical Interpretation of the Kinetic Theory*. Moscow (1886).
11. L. Boltzmann, Lectures on gas theory, Gazizdat. Tekhniko-Teor. Lit., Moscow (1956).
12. A. S. Predvoditelev, To the problem on state equation for condensed media, *Inzh. Fiz. Zh.* **5**(8), 11 (1962).
13. G. C. Maxwell, *To the Kinetic Theory of Gases*, p. 1190. Founders of the Material Kinetic Theory, Moscow (1937).
14. S. H. Burbury, On Boltzmann's law of the equality of mean kinetic energy for each degree of freedom, *Proc. Lond. Math. Soc.* **27**, 214 (1896).
15. V. A. Bubnov, On correlation function of molecules velocity distribution, *Inzh. Fiz. Zh.* **2** (1974).

LE CONCEPT D'ONDES EN THEORIE DE LA CHALEUR

Résumé—On obtient la solution fondamentale classique de l'équation de la conduction thermique sous forme d'une intégrale de Poisson en sommant des solutions particulières de similitude obtenues à l'aide d'un groupe de similitude. Ces solutions de similitude ne satisfaisant pas les conditions initiales, on obtient alors la condition intégrale à partir de laquelle la constante arbitraire de la solution générale est déterminée. On montre ensuite que la solution sous forme d'intégrale de Poisson satisfait approximativement la condition initiale sous forme de fonction discontinue en escalier approchant la fonction initiale. C'est un des inconvénients de la solution fondamentale de l'équation classique de la chaleur.

Le paradoxe de la "vitesse de propagation infinie de la chaleur" est discuté.

Analysant le comportement des isothermes et utilisant le théorème de Green, on obtient l'équation générale non-linéaire des ondes dans laquelle la vitesse de propagation des isothermes suivant la normale est utilisée comme paramètre expérimental. La relation entre cette vitesse et la diffusivité thermique est établie. Les différents cas particuliers de cette équation d'onde sont étudiés. On montre que certaines solutions de l'équation d'onde correspondent à celles de l'équation parabolique non-linéaire.

L'équation ondulatoire de la conduction thermique est alors déduite à partir de considérations de cinétique moléculaire. Le concept de temps de relaxation est généralisé en utilisant la méthode de Maxwell et en tenant compte des corrélations entre composantes d'une vitesse thermique des atomes ou molécules.

WELLENKONZEPT IN DER WÄRMETHEORIE

Zusammenfassung—Die bekannte Fundamentallösung der Wärmeleitungsgleichung in Form des Poisson-Integrals wird durch Summierung partikulärer Ähnlichkeitslösungen erhalten, die aus Ähnlichkeitsgruppen abgeleitet sind. Da diese Ähnlichkeitslösungen den Anfangsdaten nicht genügen, werden Integralbedingungen zugrundegelegt, aus denen die beliebigen Konstanten der allgemeinen Lösung bestimmt werden. Damit befriedigt die Lösung in Form des Poisson-Integrals näherungsweise die Anfangsbedingung in Form einer diskontinuierlichen Schrittfunktion als Näherung für die Anfangsfunktion. Dies ist einer der Nachteile der Fundamentallösung der klassischen Wärmeleitungsgleichung.

Das Paradoxon der unendlichen "Wärme-Fortpflanzungs-Geschwindigkeit" wird diskutiert.

Aufgrund der Analyse des isothermen Verhaltens und mit Hilfe des Greenschen Theorems wird die allgemeine nicht-lineare Wellengleichung abgeleitet, worin die Geschwindigkeit der isothermen Ausbreitung in Normalenrichtung als ein experimenteller Parameter verwendet wird. Der Zusammenhang zwischen dieser Geschwindigkeit und dem Temperaturleitvermögen wird gezeigt. Die verschiedenen partikulären Fälle dieser Wellengleichung werden untersucht. Einige der Lösungen korrespondieren mit jenen der nicht-linearen parabolischen Gleichung.

Weiterhin ist die Albeitung der Wellen-Wärmeleitungsgleichung aus der Sicht molekular-kinetischer Überlegungen gegeben. Die Vorstellung der Zeitrelaxation ist verallgemeinert unter Benutzung der Maxwell-Methode und unter Berücksichtigung der Beziehungen zwischen den Komponenten einer Wärmegeschwindigkeit von Atomen oder Molekülen.

ВОЛНОВЫЕ ПРЕДСТАВЛЕНИЯ В ТЕОРИИ ТЕПЛА

Аннотация — Известное фундаментальное решение уравнения теплопроводности в форме интеграла Пуассона получено суммированием частных автомодельных решений, полученных с помощью группы-подобия. Так как такие автомодельные решения не удовлетворяют начальным данным, то получено интегральное условие, из которого определяется произвольная константа общего решения. Далее показано, что решение в виде интеграла Пуассона удовлетворяет начальному условию приближенно в виде разрывной ступенчатой функции аппроксимирующей начальную функцию. Это один из недостатков фундаментального решения классического уравнения теплопроводности.

Обсуждается парадокс о бесконечной «скорости распространения тепла».

Анализируя поведения изотерм и пользуясь теоремой Грина, в работе дан вывод нелинейного волнового уравнения самого общего вида, в котором в качестве экспериментального параметра используется скорость распространения изотермы по нормали. Указывается связь этой скорости с коэффициентом температуропроводности. Изучаются различные частные случаи этого волнового уравнения. Показано, что некоторые решения волнового уравнения соответствуют решениям нелинейного параболического уравнения.

Далее дается вывод волнового уравнения теплопроводности из молекулярно кинетических соображений. Для этого использован метод Максвелла и, учитывая корреляцию в составляющих тепловой скорости атомов или молекул, обобщено понятие времени релаксации.